JAYNES-CUMMINGS MODEL WITH DEGENERATE ATOMIC LEVELS

V. A. Reshetov

Department of Physics, Tolyatti Pedagogical Institute, 13 Boulevard Korolyova, 445859 Tolyatti, Russia

Abstract

The Jaynes-Cummings model describing the interaction of a single linearly-polarized mode of the quantized electromagnetic field with an isolated two-level atom is generalized to the case of atomic levels degenerate in the projections of the angular momenta on the quantization axis, which is a usual case in the experiments. This generalization, like the original model, obtains the explicit solution. The model is applied to calculate the dependence of the atomic level populations on the angle between the polarization of cavity field mode and that of the laser excitation pulse in the experiment with one-atom micromaser.

The Jaynes-Cummings model [1] describes the interaction of a single linearly-polarized mode of the quantized electromagnetic field with an isolated two-level atom. The full set of states of the system atom+field is

$$|n,\alpha>=|n>\cdot|\alpha>, \quad ,n=0,1,..., \quad \alpha=b,c,$$

where n is the number of photons in the field mode, while b and c denote the upper and lower atomic levels correspondingly. This model is applied successfully to analyse the results of the experiments with one-atom micromasers (see, e.g., [2]). However, the levels of an isolated atom are degenerate in the projections of the total elctronic angular momenta on the quantization axis, so that the original Jaynes-Cummings model becomes, in general, invalid.

Now, let us take into account the degeneracy of atomic levels. Then, the full set of states of the system becomes

$$|n,J_\alpha,m_\alpha>=|n>\cdot|J_\alpha,m_\alpha>,\ n=0,1,...,\ m_\alpha=-J_\alpha,...,J_\alpha,\ \alpha=b,c,$$

where J_b and J_c are the values of the total electronic angular momenta of resonant levels, while m_b and m_c are their projections on the quantization axis - the cartesian axis Z, which is directed along the polarization vector of the field mode.

The Hamiltonian of the system may be written as

$$\hat{H}=\hat{H}_F+\hat{H}_A+\hat{V}\ , \tag{1}$$

where

$$\hat{H}_F=\hbar\omega\hat{a}^+\hat{a}$$

is a free-field Hamiltonian,

$$\hat{H}_A=\frac{1}{2}\hbar\omega_0(\hat{n}_b-\hat{n}_c)$$

is a free-atom Hamiltonian,

$$\hat{V}=-(\hat{\mathbf{D}}\hat{\mathbf{E}})$$

is an operator of field-atom interaction, while $\hat{a}^+$ and $\hat{a}$ are the operators of the creation and annihilation of photons with the frequency ω in the field mode,

$$\hat{n}_\alpha = \sum_{m_\alpha=-J_\alpha}^{J_\alpha} |J_\alpha, m_\alpha >< J_\alpha, m_\alpha|, \ \alpha = b, c,$$

are the operators of total populations of resonant atomic levels b and c, ω_0 is the frequency of the optically-allowed atomic transition $J_b \to J_c$,

$$\hat{\mathbf{E}} = \mathbf{e}\hat{a} + \mathbf{e}^*\hat{a}^+,$$

$$\mathbf{e} = \imath \mathbf{l}_z \sqrt{\frac{2\pi\hbar\omega}{V}},$$

is the electric field intensity operator, V and $\mathbf{l}_z$ being the resonator cavity volume and the unit vector of the cartesian axis Z,

$$\hat{\mathbf{D}} = \hat{\mathbf{d}} + \hat{\mathbf{d}}^+,$$

$$\hat{\mathbf{d}} = \sum_{m_b, m_c} \mathbf{d}^{J_c J_b}_{m_c m_b} \cdot |J_c, m_c >< J_b, m_b|,$$

is the dipole moment operator of the atomic transition $J_b \to J_c$, which matrix elements are defined through Wigner 3j-symbols (see, e.g., [3]):

$$(d_q)^{J_b J_c}_{m_c m_b} = d(-1)^{J_b - m_b} \begin{pmatrix} J_b & 1 & J_c \\ -m_b & q & m_c \end{pmatrix},$$

$d = d(J_b J_c)$ -being a reduced matrix element and d_q $(q = -1, 0, 1)$ - are the circular components of vector $\mathbf{d}$.

In the interaction representation

$$\hat{f}_I = \exp\left(\frac{\imath \hat{H}_0 t}{\hbar}\right) \cdot \hat{f} \cdot \exp\left(-\frac{\imath \hat{H}_0 t}{\hbar}\right),$$

where

$$\hat{H}_0 = \hbar\omega \left\{ \hat{a}^+\hat{a} + \frac{1}{2}(\hat{n}_b - \hat{n}_c) \right\},$$

the operators $\hat{a}$ and $\hat{\mathbf{d}}$ obtain the oscillating factors

$$\hat{a}_I = \hat{a} \cdot \exp(-\imath\omega t), \quad \hat{\mathbf{d}}_I = \hat{\mathbf{d}} \cdot \exp(-\imath\omega t).$$

Then, in the rotating wave approximation, when the terms oscillating with double frequences are neglected, the Hamiltonian (1) becomes

$$\hat{H}_I = \hat{H}_0 - \hbar\hat{\Omega},$$

where

$$\hat{\Omega} = \frac{\delta}{2}(\hat{n}_b - \hat{n}_c) + \imath g(\hat{a}\hat{p}^+ - \hat{a}^+\hat{p}),$$

while

$$\delta = (\omega - \omega_0)$$

is the frequency detuning,

$$g = \sqrt{\frac{2\pi d^2 \omega}{\hbar V}}$$

and

$$\hat{p} = \sum_m \alpha_m \cdot |J_c, m >< J_b, m| \ ,$$

$$\alpha_m = (-1)^{J_b - m} \begin{pmatrix} J_b & 1 & J_c \\ -m & 0 & m \end{pmatrix}.$$

From the equation

$$\frac{d\hat{\sigma}}{dt} = \frac{\imath}{\hbar}\left[\hat{\sigma}, \hat{H}\right]$$

for the system density matrix $\hat{\sigma}$ follows the equation

$$\frac{d\hat{\rho}}{dt} = \imath\left[\hat{\Omega}, \hat{\rho}\right] \tag{2}$$

for the density matrix

$$\hat{\rho} = \exp\left(\frac{\imath\hat{H}_0 t}{\hbar}\right) \cdot \hat{\sigma} \cdot \exp\left(-\frac{\imath\hat{H}_0 t}{\hbar}\right)$$

in the interaction representation. The formal solution of the equation (2) is obtained immediately

$$\hat{\rho} = \exp\left(\imath\hat{\Omega}t\right) \cdot \hat{\rho}_0 \cdot \exp\left(-\imath\hat{\Omega}t\right),$$

where $\hat{\rho}_0$ is the initial density matrix of the system.

In order to obtain the average value

$$< \hat{f} >= Tr\left(\hat{\rho}\hat{f}_I\right)$$

of any operator $\hat{f}$ it is necessary to calculate the matrix elements

$$< n, J_\alpha, m| \exp\left(\imath\hat{\Omega}t\right)|n_1, J_\beta, m_1 >,\ \alpha, \beta = b, c,$$

of the evolution operator. The explicit analytical expressions for these matrix elements may be derived with the use expansion

$$\exp\left(\imath\hat{\Omega}t\right) = \sum_{n=0}^{\infty} \frac{(\imath\hat{\Omega}t)^n}{n!}\ ,$$

since the operator

$$\hat{\Omega}^2 = \frac{\delta^2}{4}\hat{1} + g^2 \cdot \left\{\hat{R}_c\hat{n} + \hat{R}_b(\hat{n}+1)\right\},$$

where

$$\hat{R}_\beta = \sum_m \alpha_m^2 \cdot |J_\beta, m >< J_\beta, m|,\ \beta = b, c,$$

is diagonal:

$$\hat{\Omega}^2|n, J_b, m >= \Omega_{n+1,m}^2|n, J_b, m >,$$

$$\hat{\Omega}^2|n, J_c, m >= \Omega_{n,m}^2|n, J_c, m > .$$

Here

$$\Omega_{n,m} = \sqrt{\frac{\delta^2}{4} + \alpha_m^2 g^2 n}. \tag{3}$$

So, the matrix elements of the evolution operator are:

$$< n, J_b, m| \exp\left(\imath\hat{\Omega}t\right)|n_1, J_b, m_1 >=$$

$$\delta_{n,n_1}\delta_{m,m_1}\left\{\cos\left(\Omega_{n+1,m}t\right) + \frac{\imath\delta}{2\Omega_{n+1,m}}\sin\left(\Omega_{n+1,m}t\right)\right\}\ , \tag{4}$$

$$< n, J_c, m| \exp\left(\imath\hat{\Omega}t\right)|n_1, J_c, m_1 >=$$

$$\delta_{n,n_1}\delta_{m,m_1}\left\{\cos\left(\Omega_{n,m}t\right)-\frac{\imath\delta}{2\Omega_{n,m}}\sin\left(\Omega_{n,m}t\right)\right\}\ , \tag{5}$$

$$< n, J_b, m|\exp\left(\imath\hat{\Omega}t\right)|n_1, J_c, m_1 >=$$

$$-\delta_{n+1,n_1}\delta_{m,m_1}g\alpha_m\sqrt{n+1}\cdot\frac{\sin\left(\Omega_{n+1,m}t\right)}{\Omega_{n+1,m}}\ . \tag{6}$$

In the experiment [2] the average total population

$$n_b = Tr\left\{\hat{n}_b\exp\left(\imath\hat{\Omega}T\right)\rho_0\exp\left(-\imath\hat{\Omega}T\right)\right\}$$

of the upper resonant level b after the atom passes through the resonant cavity, where T is the time of interaction, was detected. As follows from (4)-(6),

$$n_b = \sum_{n,m} f_{nn}n^b_{mm}\left\{\cos^2\left(\Omega_{n+1,m}T\right)+\frac{\delta^2}{4\Omega^2_{n+1,m}}\sin^2\left(\Omega_{n+1,m}T\right)\right\}\ , \tag{7}$$

where the atomic and field subsystems at the initial instant of time, when the atom enters the cavity, are independent and the initial density matrix of the system is

$$\hat{\rho}_0 = \hat{\rho}^A_0\cdot\hat{\rho}^F_0\ ,$$

while

$$\hat{\rho}^A_0 = \sum_{m,m_1} n^b_{mm_1}\cdot|J_b, m >< J_b, m_1|\ ,$$

$$\hat{\rho}^F_0 = \sum_{n,n_1} f_{nn_1}\cdot|n >< n_1|\ .$$

The cavity temperature in [2] was low, so that the initial field may be considered to be in its vacuum state:

$$f_{n,n_1} = \delta_{n,0}\delta_{n_1,0}.$$

Then, in case of exact resonance $\delta = 0$ the equation (7) simplifies to

$$n_b = \sum_m n^b_{mm}\cos^2\left(\theta_m\right)\ ,\ \theta_m = \alpha_m gT\ . \tag{8}$$

Here n^b_{mm} is the initial population of the Zeeman sublevel m of the upper level b . The resonant levels b and c in the experiment [2] were the Rydberg states of the rubidium atom with the angular momenta $J_b = 3/2$ and $J_c = 3/2$ or $J_c = 5/2$. The upper level b was excited from the ground state a with the angular momentum $J_a = 1/2$ by the linearly-polarized laser pulse. The evolution of the atomic density matrix under the action of the excitation pulse in the rotating-wave approximation is desribed by the equation

$$\frac{d\hat{\rho}^A}{dt} = \frac{\imath}{\hbar}\left[\hat{\rho}^A, \hat{V}_e\right]\ , \tag{9}$$

where

$$\hat{V}_e = -(\hat{\mathbf{d}}^+_e\mathbf{e}_e + \hat{\mathbf{d}}_e\mathbf{e}^*_e)$$

is the interaction operator of an atom with the cohernt resonant laser field,

$$\hat{\mathbf{d}}_e = \sum_{m_b,m_a}(\mathbf{d}_e)^{J_aJ_b}_{m_am_b}\cdot|J_a, m_a >< J_b, m_b|,$$

is the dipole moment operator of the atomic transition $J_b \to J_a$, $\mathbf{e}_e = e_e \mathbf{l}$ is the slowly-varying amplitude of laser field, $\mathbf{l}$ is its unit polarization vector, which constitutes the angle ψ with the polarization of the cavity field mode:

$$l_q = \cos\psi\delta_{q,0} + \frac{1}{\sqrt{2}}\sin\psi(\delta_{q,-1} - \delta_{q,1}) \ .$$

For purposes of simplicity we shall consider the exciting pulses with small areas

$$\theta_e = \frac{|d_e|}{\hbar}\int_0^{T_e} e_e(t)dt \ \ll 1 \tag{10}$$

(though in case of transition $3/2 \to 1/2$ in the experiment [1] the following results do not depend on the exciting pulse area), $d_e = d(J_b J_a)$ is a reduced matrix element of the dipole moment operator for the transition $J_b \to J_a$, T_e is the exciting pulse duration. Under the limitation (10) we obtain from (9) the density matrix of an atom (renormalized to unity trace)

$$\hat{\rho}_0^A = \frac{(\hat{\mathbf{d}}_e^+\mathbf{l})\hat{\rho}_{in}^A(\hat{\mathbf{d}}_e\mathbf{l})}{Tr\left\{(\hat{\mathbf{d}}_e^+\mathbf{l})\hat{\rho}_{in}^A(\hat{\mathbf{d}}_e\mathbf{l})\right\}} \tag{11}$$

at an instant when it enters the cavity. Here

$$\hat{\rho}_{in}^A = \frac{1}{(2J_a+1)}\sum_m |J_a, m><J_a, m|$$

is the initial equilibrium atomic density matrix before the incidence of the exciting pulse. As follows from (11) the Zeeman sublevel populations in (8) are

$$n_{mm}^b =< J_b, m|\hat{\rho}_0^A|J_b, m >= a_m\cos^2\psi + b_m\sin^2\psi \ ,$$

where

$$a_m = 3\begin{pmatrix} J_b & 1 & J_a \\ -m & 0 & m \end{pmatrix}^2 \ ,$$

$$b_m = \frac{3}{2}\left\{\begin{pmatrix} J_b & 1 & J_a \\ -m & -1 & m+1 \end{pmatrix}^2 + \begin{pmatrix} J_b & 1 & J_a \\ -m & 1 & m-1 \end{pmatrix}^2\right\} \ .$$

In case of transitions $J_b = 3/2 \to J_a = 1/2$

$$n_{-1/2,-1/2}^b = n_{1/2,1/2}^b = \frac{1}{2} - \frac{3}{8}\sin^2\psi \ ,$$

$$n_{-3/2,-3/2}^b = n_{3/2,3/2}^b = \frac{3}{8}\sin^2\psi \ ,$$

and the total population (8) of the upper level after the atom leaves the cavity is

$$n_b = \left(1 - \frac{3}{4}\sin^2(\psi)\right)\cos^2(\theta) + \frac{3}{4}\sin^2(\psi)\cos^2(3\theta) \ ,$$

$$\theta = \frac{gT}{2\sqrt{15}} \ ,$$

for the transitions $J_b = 3/2 \to J_c = 3/2$ and

$$n_b = \left(1 - \frac{3}{4}\sin^2(\psi)\right)\cos^2(\theta) + \frac{3}{4}\sin^2(\psi)\cos^2\left(\sqrt{\frac{3}{2}}\theta\right) \ ,$$

$$\theta = \frac{gT}{\sqrt{10}} \ ,$$

for the transitions $J_b = 3/2 \rightarrow J_c = 5/2$.

The atom behaves like a two-level system - the population

$$n_b = \cos^2(\theta)$$

oscillates with a single Rabi frequency - only in case when the polarizations of the exciting laser pulse and of the cavity field mode coincide -
$\psi = 0$, otherwise the oscillations with more than one Rabi frequencies appear.

So, the Jaynes-Cummings model generalized to the case of the atomic levels degenerate in the projections of the angular momenta on the quantization axis is a useful tools for the description of the polarization properties of one-atom micromasers.

References

[1] Jaynes E T, Cummings F W 1963 *Proc. IEEE* **51** 89
[2] Walther H 1995 *Ann.N.Y.Acad.Sci.* **755** 133
[3] Sobelman I I 1972 *Introduction to the Theory of Atomic Spectra* (New York:Pergamon)